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Abstract

We present here some algebraic formulas enabeling to define a k-automaton $\mathcal{A}_2$ from a given k-automaton $\mathcal{A}_1$ such that the behaviour of $\mathcal{A}_2$ is the behaviour of $\mathcal{A}_1$ after erasure of a given set of letters. This procedure contains as particular case the algebraic elimination of ε-transitions. The time complexity of this process is evaluated. In the case of well-known semirings (boolean and tropical) the closure is computed in $O(n^3)$. When k is a ring, the complexity can be more finely tuned.

0 Introduction

Automata with multiplicities (or weighted automata) are a versatile class of transition systems which can modelize as well classical (boolean), stochastic, transducer automata and be applied to various purposes such as image compression, speech recognition, formal linguistic (and automatic treatment of natural languages too) and probabilistic modelling. For generalities over automata with multiplicities see [1] and [10], problems over identities and decidability results on these objects can be found in [13], [12] and [11]. One among many operations which can be applied on the k-automata is the elimination of a-transitions where a belongs to a subset of the alphabet A. The aim of this work is indeed to give algebraic formulas for the erasure of a-transitions from a k-automata $\mathcal{A}$ where $a \in Z = \{a_1, a_2, \cdots, a_m\}$ with $a_i \in A$. We give a procedure based on Lazard's elimination in the case where the set of a-transitions is globally nilpotent. The result depends only on the computation of the star M^* of transition matrix $M_Z = M_{a_1} + \cdots + M_{a_m}$. The time complexity of this process is evaluated. In the case of well-known semirings (boolean and tropical) the closure is computed in $O(n^3)$. When k is a ring, the complexity can be more finely tuned.

The structure of the paper is the following. Section 1 is devoted to recalling general points about functions on the free monoid (i.e. noncommutative series) and to the problem of monomial transformations (i.e. transformation of the words within the series). In section 2 we expound the star problem in full generality. Then, after introducing (in Section 3) the notion of k-automaton, we present (in Section 4 and 5) our principal result which is an algebraic method of erasure of given letter-transitions. In Section 6, we give more examples and another complexity result.
A conclusion section ends the paper.

1 Word transformations of series

In the sequel, we discuss the possibility of changing the monomials of a noncommutative series into other monomials. As we will deal with finiteness properties from now on, all the alphabets will be supposed finite.
A series (noncommutative) S is just a mapping $A^* \mapsto k$, where A is an alphabet (i.e. a set) of (non-commuting) variables and k some set of coefficients (or scalars). When k is endowed with some operations, it induces structure to the set of series too. The most popular, for k, is the structure of a *semiring*. For generalities on semirings one can consult [2, 7, 10] or
`http://mathworld.wolfram.com/Semiring.html`.

*Also in LIPN, Université Paris XIII.

In the sequel, however, semirings will always be supposed with neutrals. Indeed, here, a semiring $(k, \oplus, \otimes, 0_k, 1_k)$ is a set together with two laws and their neutrals. More precisely $(k, \oplus, 0_k)$ is a commutative monoid with 0_k as neutral and $(k, \otimes, 1_k)$ is a monoid with 1_k as neutral. The product is distributive with respect to the addition and zero is an annihilator ($0_k \otimes x = x \otimes 0_k = 0_k$) [7]. The boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$ and the tropical semiring $\mathbb{T} = (\mathbb{R}_+ \cup \{\infty\}, \min, +, \infty, 0)$ are well-known examples of semirings.

A series $S : A^* \mapsto k$ can conveniently be denoted as $S = \sum_{w \in A^*} \langle S|w\rangle w$, where $\langle S|w\rangle = S(w)$ is the image of w by S.

Now, after [1], we will say that a family of series $(S_i)_{i \in I}$ is *summable* iff, for all $w \in A^*$ the set

$$\{j \in I | \langle S_j|w\rangle \neq 0\} \tag{1}$$

is finite. In this case, the sum $\sum_{i \in I} S_i$ is the series T such that

$$\langle T|w\rangle = \sum_{i \in I} \langle S_i|w\rangle \tag{2}$$

for all w. Hence, by definition,

$$\sum_{i \in I} S_i := \sum_{w \in A^*} \left(\sum_{i \in I} \langle S_i|w\rangle \right) w. \tag{3}$$

When I has two elements, one recovers the definition of the usual sum by

$$S + T = \sum_{w \in A^*} \left(\langle S|w\rangle + \langle T|w\rangle\right) w \tag{4}$$

it is straightforward to check that, for this law, the neutral is the null function.

Concatenation, extended by the Cauchy product, will read

$$S.T = \sum_{w \in A^*} \left(\sum_{uv = w} \langle S|u\rangle \langle T|v\rangle \right) w. \tag{5}$$

The neutral is the Dirac unital charge supported by $\varepsilon \in A^*$, the empty word. As this causes no confusion, this function will still be denoted by ε and more generally a Dirac unital charge supported by $w \in A^*$ will be identified to w.

Endowed with the two preceding laws and neutrals k^{A^*} is a semiring denoted $k\langle\langle A\rangle\rangle$.

The *support* of a series is the set of words where it takes non zero values i.e.

$$\mathrm{supp}(S) = \{w \in A^* | \langle S|w\rangle \neq 0\} \tag{6}$$

A series with finite support is called a (noncommutative) polynomial. The set of these series is a subsemiring of $k\langle\langle A\rangle\rangle$ denoted $k\langle A\rangle$.

Let now B be another alphabet. Every (set-theoretical) mapping $\Phi : A \mapsto B^*$ can be uniquely extended to A^* as a morphism of monoids by

$$\Phi(a_1 a_2 \cdots a_n) = \Phi(a_1)\Phi(a_2) \cdots \Phi(a_n) \tag{7}$$

even if $n = 0$ that is to say $\Phi(\varepsilon_{A^*}) = \varepsilon_{B^*}$. In the same way Φ is extended by linearity to a morphism $\Phi_{\mathbf{alg}} : k\langle A\rangle \mapsto k\langle B\rangle$ by

$$\Phi_{\mathbf{alg}} \left(\sum_{w \in A^*} \langle S|w\rangle w \right) = \sum_{w \in A^*} \langle S|w\rangle \Phi(w). \tag{8}$$

This is well-defined as the sums here are, in fact, finite.

At this point, the question which must be considered here is:

How much $\Phi_{\mathbf{alg}}$ can be extended to $k\langle\langle A\rangle\rangle$?

The first attempt can be performed using formula (8). In this case we must ask that the family of monomials of the second member, namely

$$\Big(\langle S|w\rangle \Phi(w) \Big)_{w \in A^*} \tag{9}$$

be summable. Denoting classically the preimage of a word $w \in B^*$ by $\Phi^{-1}(w)$, one can check that the summability is equivalent to condition (**FF**):

(**FF**) For all $w \in A^*$, the set $\mathrm{supp}(S) \cap (\Phi^{-1}(w))$ is finite.

If the formal series S satisfies (**FF**), we say that it is Φ-finite. The set of Φ-finite series in $k\langle\langle A\rangle\rangle$ is denoted $k\langle\langle A\rangle\rangle_{\Phi-finite}$. We denote by $\alpha(?)$ and $(?)\alpha$ the left and right external product respectively, and we show now that it is a subalgebra of $k\langle\langle A\rangle\rangle$.

Theorem 1 *The set $k\langle\langle A\rangle\rangle_{\Phi-finite}$ is closed by $+$, $\cdot$, $\alpha(?)$ and $(?)\alpha$ (the right and left scaling).*

Proof. As $\mathrm{supp}(S_1+S_2) \subseteq \mathrm{supp}(S_1) \cup \mathrm{supp}(S_2)$, $\mathrm{supp}(\alpha S_1) \subseteq \mathrm{supp}(S_1)$ and $\mathrm{supp}(S_1\alpha) \subseteq \mathrm{supp}(S_1)$ for $S_1, S_2 \in k\langle\langle A\rangle\rangle$ and $\alpha \in k$, the stability is shown for $+$, $\alpha(?)$ and $(?)\alpha$.
Now, for the Cauchy product, one has:

$$\mathrm{supp}(S_1S_2) \subseteq \mathrm{supp}(S_1)\mathrm{supp}(S_2). \tag{10}$$

Then, for every $v \in B^*$ one has,

$$\mathrm{supp}(S_1S_2) \cap \Phi^{-1}(v) \subseteq$$

$$\bigcup_{v=v_1v_2} (\mathrm{supp}(S_1) \cap \Phi^{-1}(v_1))(\mathrm{supp}(S_2) \cap \Phi^{-1}(v_2))$$

the right hand side being a finite set if $S_1, S_2 \in k\langle\langle A\rangle\rangle_{\Phi-finite}$. □

In case S is Φ-finite, we can set

$$\Phi(S) = \sum_{w\in A^*} \langle S|w\rangle \Phi(w) \tag{11}$$

which extends Φ.

Remark 1 *i) Every polynomial is Φ-finite.*
ii) The star of a series S^* need not be Φ-finite even if S is. The simplest example is provided by

$$A = B = \{a\},\ \Phi(a) = \varepsilon,\ S = a \tag{12}$$

then S is Φ-finite and not S^*.

Next we show that $\Phi : k\langle\langle A\rangle\rangle_{\Phi-finite} \mapsto k\langle\langle A\rangle\rangle$ is a morphism of algebras.

Theorem 2 *For any $S, T \in k\langle\langle A\rangle\rangle_{\Phi-finite}$,*

$$\Phi(S+T) = \Phi(S) + \Phi(T),\ \Phi(ST) = \Phi(S)\Phi(T),$$
$$\Phi(\alpha S) = \alpha\Phi(S),\ \Phi(S\alpha) = \Phi(S)\alpha$$

2 The star calculus over a semiring

Generalities over semirings can be found in [10]. In the sequel, the star of a scalar is introduced by the following definition:

Definition 1 *Let $x \in k$, the scalar y is a right (resp. left) star of x if and only if $x \otimes y \oplus 1_k = y$ (resp. $y \otimes x \oplus 1_k = y$).*

If $y \in k$ is a left and right star of $x \in k$, we say that y is a star for x and we write $y = x^{\circledast}$.

Example 1

1. For $k = \mathbb{C}$ (and more generally any field), any complex number $x \neq 1$ has a unique star which is $y = (1-x)^{-1}$. In the case $|x| < 1$, we observe easily that $y = 1 + x + x^2 + \cdots$.

2. Let k be the ring of all linear operators ($\mathbb{R}[x] \to \mathbb{R}[x]$). Let X and Y_α defined by $X(x^0) = 1$, $X(x^n) = x^n - nx^{n-1}$ with $n > 0$ and $Y_\alpha(x^n) = (n+1)^{-1}x^{n+1} + \alpha$ with $\alpha \in \mathbb{R}$. Then $XY_\alpha + 1 = Y_\alpha$ and there exists an infinite number of solutions for the right star (which is not a left star if $\alpha \neq 0$). This example explains the expressions right and left star.

3. For $k = \mathbb{T}$ (tropical semiring), any number x has a unique star $y = 0$.

4. (Star of a series) Let $S \in k\langle\langle A\rangle\rangle$. We say that the series is *proper* if the coefficient of the empty word is zero $\langle S|\varepsilon\rangle = 0$. The family $(S^i)_{i\in\mathbb{N}}$ is summable and the star is given by $S^* = \varepsilon + S + S^2 + \cdots$. More generally, the star $(?)^*$ of a noncommutative formal series is well-defined if and only if the star of $\langle S|\varepsilon\rangle$ exists [10, 1]. Then $S^* = \alpha^{\circledast}(S_0\alpha^{\circledast})^*$ if $\alpha = \langle S|\varepsilon\rangle$ and $S = \alpha + S_0$.

   The set $k\langle\langle A\rangle\rangle^{\mathrm{rat}}$ is the closure of $k\langle A\rangle$ by the sum, the Cauchy product and the star.

We can observe that if the opposite $-x$ of x exists then right (resp. left) stars of x are the right (resp. left) inverses of $(1\oplus(-x))$ and conversely. Any right star $x^{\circledast_r}$ equals any left star $x^{\circledast_l}$ as $x^{\circledast_l} = x^{\circledast_l} \otimes ((1 \oplus (-x)) \otimes x^{\circledast_r}) = (x^{\circledast_l} \otimes (1 \oplus (-x))) \otimes x^{\circledast_r} = x^{\circledast_r}$. Thus, in this case, the star is unique.

If n is a positive integer then the set $k^{n\times n}$ of square matrices with coefficients in k has a natural structure of semiring with the usual operations (sum and product). The (right) star of $M \in k^{n\times n}$ (when there exists) is a solution of the equation $MY + 1_{n\times n} = Y$ (where $1_{n\times n}$ is the identity matrix).
Let $M \in k^{n\times n}$ be given by

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $a_{11} \in k^{p\times p}$, $a_{12} \in k^{p\times q}$, $a_{21} \in k^{q\times p}$ and $a_{22} \in k^{q\times q}$ such that $p+q=n$. Let $N \in k^{n\times n}$ given by

$$N = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with

$$A_{11} = (a_{11} + a_{12}{a_{22}}^* a_{21})^* \quad (13)$$
$$A_{12} = {a_{11}}^* a_{12} A_{22} \quad (14)$$
$$A_{21} = {a_{22}}^* a_{21} A_{11} \quad (15)$$
$$A_{22} = (a_{22} + a_{21}{a_{11}}^* a_{12})^* \quad (16)$$

Theorem 3 *If the right hand sides of the formulas (13), (14), (15) and (16) are defined, the matrix M admits N as a right star.*

Proof. We have to show that N is a solution of the equation $My + 1_{n\times n} = y$. By computation, one has MN+1=

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} I & 0_{p\times q} \\ 0_{q\times p} & I' \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}A_{11} + a_{12}A_{21} + I & a_{11}A_{12} + a_{12}A_{22} \\ a_{21}A_{11} + a_{22}A_{21} & a_{21}A_{12} + a_{22}A_{22} + I' \end{pmatrix}$$

where $0_{p\times q}$ is the zero matrix in $k^{p\times q}$, $I = 1_{p\times p}$ and $I' = 1_{q\times q}$. We verify the relations (13), (14), (15) and (16) by:

$$\begin{aligned} a_{11}A_{11} + a_{12}A_{21} + I &= a_{11}A_{11} + a_{12}a_{22}^* a_{21}A_{11} + I \\ &= A_{11}(a_{11} + a_{12}{a_{22}}^* a_{21}) + I \\ &= A_{11} \end{aligned}$$

$$\begin{aligned} a_{11}A_{12} + a_{12}A_{22} &= a_{11}{a_{11}}^* a_{12}A_{22} + a_{12}A_{22} \\ &= (a_{11}{a_{11}}^* + 1)a_{12}A_{22} \\ &= {a_{11}}^* a_{12}A_{22} \\ &= A_{12} \end{aligned}$$

$$\begin{aligned} a_{21}A_{11} + a_{22}A_{21} &= a_{21}A_{11} + a_{22}{a_{22}}^* a_{21}A_{11} \\ &= (1 + a_{22}{a_{22}}^*)a_{21}A_{11} \\ &= {a_{22}}^* a_{21}A_{11} \\ &= A_{21} \end{aligned}$$

$$\begin{aligned} a_{21}A_{12} + a_{22}A_{22} + I' &= a_{21}{a_{11}}^* a_{12}A_{22} + a_{22}A_{22} + I' \\ &= (a_{22}a_{21}{a_{11}}^* a12)A_{22} + I' \\ &= A_{22} \end{aligned}$$

□

Similar formulas can be stated in the case of the left star. The matrix N is the left star of M with

$$\begin{aligned} A_{11} &= (a_{11} + a_{12}{a_{22}}^* a_{21})^* \\ A_{12} &= A_{11}a_{12}{a_{22}}^* \\ A_{21} &= A_{22}a_{21}{a_{11}}^* \\ A_{22} &= (a_{22} + a_{21}{a_{11}}^* a_{12})^* \end{aligned}$$

In [8] and [17], similar formulas are expressed for the computation of the inverse of matrices when k is a division ring (it can be extended in the case of rings).

The formulas described above are valid with matrices of any size with any block partitionning. Matrices of even size are often, in practice, partitionned into square blocks but, for matrices with odd dimensions, the approach called dynamic peeling is applied [9]. More specifically, let $M \in k^{n\times n}$ a matrix given by

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where $n \in 2\mathbb{N}+1$. The dynamic peeling consists of cutting out the matrix in the following way: a_{11} is a $(n-1)\times(n-1)$ matrix, a_{12} is a $(n-1)\times 1$ matrix, a_{21} is a $1\times(n-1)$ matrix and a_{22} is a 1×1 matrix.

Theorem 4 *Let k be a semiring. The right (resp. left) star of a matrix of size $n \in \mathbb{N}$ can be computed in $O(n^\omega)$ operations with:*

- *$\omega \leq 3$ if k is not a ring,*
- *$\omega \leq 2.808$ if k is a ring,*
- *$\omega \leq 2.376$ if k is a field.*

Proof. For $n = 2^m \in \mathbb{N}$, let T_m^+ , $T_m^\times$ and T_m^* denote the number of operations $\oplus$, $\otimes$ and $\circledast$ in k that the addition, the multiplication and the star of matrix respectively perform on input of size n . Then

$$T_0^* = 1$$
$$T_m^* = 2T_{m-1}^+ + 8T_{m-1}^\times + 4T_{m-1}^*$$

by Theorem 3, for arbitrary n , we add some zeroes at the matrix. If k is a ring, using Strassen's algorithm for the matrix multiplication [20], it is known that at most $16n^{\log_2(7)}$ operations are necessary. If k is a field, using the Coppersmith and Winograd's algorithm [3], it is known that at most $16n^{2.376}$ operations are necessary. □

The actual running time complexity for the computation of the right (resp. left) star of a matrix depends on $T_\oplus$, $T_\otimes$ and $T_\circledast$, but it depends also on the representation of coefficients in machine. In the case $k = \mathbb{Z}$ for example, the multiplication of two integers is computed in $\mathrm{O}(m \log(m) \log(\log(m)))$, using FFT if m bits is necessary [19].

Theorem 5 *The space complexity of the right (resp. left) star of a matrix of size $n \in \mathbb{N}$ is* $\mathrm{O}(n^2 \log(n))$.

Proof. For $n = 2^m \in \mathbb{N}$ and k a semiring, let E_m^* denote the space complexity of operation $*$ that the star of matrix perform on input of size n. Then

$$E_0^* = 1$$
$$E_m^* = 12 \cdot 2^{2m-1} + 4E_{m-1}^*$$

□

The running of the algorithm needs the reservation of memory spaces for the result matrix (the star of the input matrix) and for intermediate results stored in temporary locations.

To end, we complete the picture about the morphism Φ defined in (2) by showing that it carries the star.

Theorem 6 *Let $S \in k\langle\langle A\rangle\rangle_{\Phi-finite}$ such that $\Phi(S^*) \in k\langle\langle B\rangle\rangle_{\Phi-finite}$ then*

$$\Phi(S^*) = \big(\Phi(S)\big)^*. \tag{17}$$

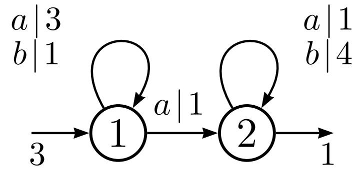

Figure 1: A $\mathbb{N}$-automaton

Proof. One has $S^* = \varepsilon + SS^*$ then $\Phi(S^*) = \varepsilon + \Phi(S)\Phi(S^*)$. □

Remark 2 *A Φ-finite series need not be rational as shown by the example:*

$$A = B = \{a, b\},\ \Phi(a) = \Phi(b) = a,\ S = \sum_{|u|_a = |u|_b} u \tag{18}$$

where S is not rational but Φ-finite.

We do not know under which conditions of Φ, the image of a rational series is rational (see below for an example of such Φ).

3 Automata with multiplicities

Let A be a finite alphabet. A weighted automaton (or linear representation) of dimension n on A with multiplicities in k is a triplet (λ, μ, γ) where:

- $\lambda \in k^{1\times n}$ (**the input vector**),
- $\mu : A \to k^{n\times n}$ (**the transition function**),
- $\gamma \in k^{n\times 1}$ (**the output vector**).

Such automaton is usually drawn by a directed valued graph (see Figure 1). A transition $(i, a, j) \in \{1, \dots, n\} \times A \times \{1, \dots, n\}$ connects the state i with the state j . Its weight is $\mu(a)_{ij}$. The weight of the initial (final) state i is λ_i (respectively γ_i). The mapping μ induces a morphism of monoid from A^* to $k^{n\times n}$. The behaviour of the weighted automaton $\mathcal{A}$ belongs to $k\langle\langle A\rangle\rangle$. It is defined by:

$$\mathrm{behaviour}(\mathcal{A}) = \sum_{u\in A^*} (\lambda\mu(u)\gamma)u.$$

More precisely, the weight $\langle \mathrm{behaviour}(\mathcal{A}), u\rangle$ of the word u in the formal series behaviour($\mathcal{A}$) is the weight of u for the k-automaton $\mathcal{A}$ [2]).

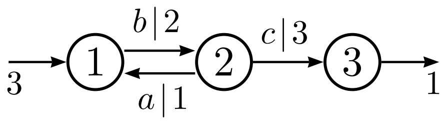

Figure 2: A $\mathbb{N}$ $\{b, c\}$-automaton

Example 2 The behaviour of the automaton $\mathcal{A}$ of Figure 1 is

$$\text{behaviour}(\mathcal{A}) = \sum_{u,v \in A^*} 3^{|u|_a+1} 4^{|v|_b} uav$$

Let $u = ab$. Then, its weight in $\mathcal{A}$ is:

$$\lambda\mu(u)\gamma = \lambda\mu(a)\mu(b)\gamma$$
$$= \begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 12.$$

Let $k\langle\langle A \rangle\rangle^{\text{rec}}$ be the set of series which are the behaviour of some weighted automata. The celebrated Theorem [18] of Schützenberger says that:

$$k\langle\langle A \rangle\rangle^{\text{rec}} = k\langle\langle A \rangle\rangle^{\text{rat}}.$$

A k Z-automaton $\mathcal{A}_Z$ is a k-automaton over an alphabet with a subset of distinguished letters $A_Z = B \cup Z$ (see Figure 2). The letters of Z are understood as ε-transitions.

Example 3 In Figure 2, the behaviour is $18b\left(\sum_{i\in\mathbb{N}} 2^i(ab)^i\right)c = 18b(2ab)^*c$.

4 Algebraic elimination

Let $Z \subseteq A$, $B = A - Z$ and Φ be the morphism from A^* to B^* defined by

$$\begin{cases} \Phi(x) = x & \text{if } x \in B, \\ \Phi(x) = \varepsilon & \text{otherwise.} \end{cases}$$

We remark that the set of antecedents of $u = a_1a_2\ldots a_n \in A^*$ by Φ can be written $\Phi^{-1}(u) = Z^*a_1Z^*a_2\cdots Z^*a_nZ^*$ (this is actually an instance of Lazard's elimination theorem [15]). For $S \in k\langle\langle A_\varepsilon \rangle\rangle$, define

$$\Phi(S) = \sum_{u\in A^*} (\sum_{\Phi(v)=u} \langle S|v\rangle)u.$$

if $\sum_{\Phi(v)=u} \langle S|v\rangle$ is defined.

Elimination of letter-transitions is the art of removing, in a given automaton $\mathcal{A}_1$, the transitions bearing letters choosen in a given subset, say Z, of an alphabet A so as to get a modified automaton $\mathcal{A}_2$ with the behaviour. The first example where this is possible is when the transition matrices $\mu(z)$ with $z \in Z$ are (globally) nilpotent, it means that there exists a bound l such that

$$(w \in Z^* \text{and} |w| \geq l \Longrightarrow \mu(w) = 0). \quad (19)$$

Proposition 1 *Let $\mathcal{A}_Z = (\lambda, \mu, \gamma)$ be a weighted Z-automaton. If $\mu(Z)$ is (globally) nilpotent then behaviour($\mathcal{A}_Z$) satisfies* (**FF**) .

Proof. Let $w = a_1a_2\cdots a_n$ one has

$$\text{supp}(S)\cap\Phi^{-1}(w) = \text{supp}(S)\cap\big(Z^{<l}a_1\cdots Z^{<l}a_nZ^{<l}\big)$$

where $Z^{<l}$ stands for $\varepsilon + Z + Z^2 + \cdots Z^{l-1}$, as if $w = z^{n_0}a_1z^{n_1}a_2\cdots z^{n-1}a_nz^n$ with one $n_j \geq l$, $\langle S|w\rangle = 0$. □

Remark 3 *It can happen that the sum of the transitions $\mu(z)$ is nilpotent but no of the individual matrices. Let, for example, take $Z = \{z_1, z_2\}$, $A = Z \cup \{a\}$ and*

$$\mu(z_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \mu(z_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\lambda = \begin{pmatrix} 1 & 0 \end{pmatrix}, \ \gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (20)$$

*The family does not fulfill (**FF**) as the sequence z_1^{2n} is in supp(S) $\cap\, \Phi^{-1}(\varepsilon)$.*

5 Examples and discussion

In the previous paragraph, we have given an algebraic method to eliminate the Z -transitions from a weighted Z -automaton $\mathcal{A}_1$ with behaviour S. The result is a weighted automaton with $\Phi(S)$ as behaviour if behaviour($\mathcal{A}_2$) $= S$. The proposition below gives the complexity in case when Z reduces to a single letter $\tilde{\varepsilon}$.

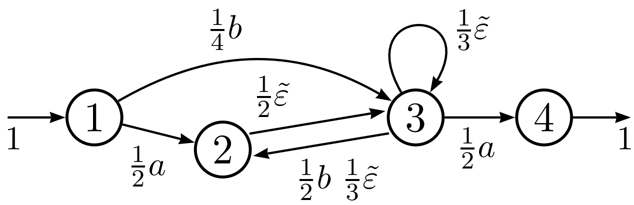

Figure 3: A $\mathbb{Q}$ ε-automaton

Proposition 2 *Let k be a semiring. The elimination of ε-transitions is computed in $O((|A|+1)\times n^{\omega})$ if n is the dimension of the weighted ε-automaton.*

Proof. First we compute the matrix $\mu(Z)^*$. Then set $\lambda' = \lambda$, $\gamma' = \mu_{\tilde{\varepsilon}}^*\gamma$ and $\mu'(a) = \mu_{\tilde{\varepsilon}}^*\mu(a)$ for each letter $a \in A$. □

Remark 4 One could also with the same result set $\lambda' = \lambda\mu_{\tilde{\varepsilon}}^*$, $\mu'(a) = \mu(a)\mu_{\tilde{\varepsilon}}^*$ for each letter $a \in A$ and $\gamma' = \gamma$.

In the next example, we will apply our algebraic method on a $\mathbb{Q}$ ε-automaton.

Example 4 The linear representation of Figure 3 is:

$$\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},\ \mu_{\tilde{\varepsilon}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mu(a) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mu(b) = \begin{pmatrix} 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } \gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

By computation:

$$\lambda' = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma' = \mu_{\tilde{\varepsilon}}^*\gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\quad \mu_{\tilde{\varepsilon}}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{4}{3} & 1 & 0 \\ 0 & \frac{2}{3} & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

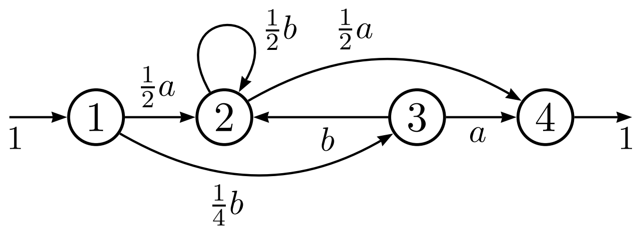

Figure 4: A $\mathbb{Q}$-automaton

$$\mu'(a) = \mu_{\tilde{\varepsilon}}^*\mu(a) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } \mu'(b) = \mu_{\tilde{\varepsilon}}^*\mu(b) = \begin{pmatrix} 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The resulting automaton is presented in Figure 4 and its linear representation is $(\lambda', \mu', \gamma')$.

6 Conclusion

Algebraic elimination for Z-automata has been presented which reduces to the problem of removing the ε-transitions when Z consists of a single letter $\tilde{\varepsilon}$.
The problem of removing the ε-transitions is originated from generic ε-removal algorithm for weighted automata [16] using Floyd-Warshall and generic single-source shortest distance algorithms. Here, we have the same objective but the methods and algorithms are different. In [16], the principal characteristics of semirings used by the algorithm as well as the complexity of different algorithms used for each step of the elimination are detailed. The case of acyclic and non acyclic automata are analysed differently. Our algorithm here works with any semiring and the complexity is unique for the case of acyclic or non acyclic automata. It is more efficient when the considered semiring is a ring.
The domain of validity of this algorithm is much more extended than the domain of the equivalence, therefore analytic elimination of transitions will be the subject of a forthcoming work.

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